Mundtlig eksamen

Jeg vil her bevise nogle påstande, således at det fremover vil være noget mere ligetil at regne med komplekse tal.

Sætning
Lad $z, w \in \mathbb{C}$ være givet ved $z = a + bi$ og $w = c + di$, så er

(i) $\overline{z+w} = \overline{z} + \overline{w}$,

(ii) $\overline{z-w} = \overline{z} - \overline{w}$,

(iii) $\overline{zw} = \overline{z}\,\overline{w}$,

(iv) $\overline{z/w} = \overline{z}/\overline{w}$,

(v) $\overline{\overline{z}} = z$,

(vi) $|zw| = |z||w|$,

(vii) $|z/w| = |z|/|w|$,

(viii) $\arg(zw) = \arg(z) + \arg(w)$,

(ix) $\arg(z/w) = \arg(z) - \arg(w)$,

hvor det i (iv), (vii) og (ix) antages, at $w \neq 0$.

Bevis
Ved brug af identiteterne $i^2 = -1$, $|z|^2 = a^2 + b^2$ og $z = |z|(\cos(\theta) + \sin(\theta)i)$, fås følgende:

$$\begin{aligned}\overline{z+w} &= \overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i} = (a+c)-(b+d)i \\ &= (a-bi)+(c-di) = \overline{z}+\overline{w},\end{aligned}$$

$$\begin{aligned}\overline{z-w} &= \overline{(a+bi)-(c+di)} = \overline{(a-c)+(b-d)i} = (a-c)-(b-d)i \\ &= (a-bi)-(c-di) = \overline{z}-\overline{w},\end{aligned}$$

$$\begin{aligned}\overline{zw} &= \overline{(a+bi)(c+di)} = \overline{ac+adi+bci+bdi^2} = \overline{(ac-bd)+(ad+bc)i} \\ &= (ac-bd)-(ad+bc)i = ac-adi-bci+bdi^2 = (a-bi)(c-di) = \overline{z}\,\overline{w},\end{aligned}$$

$$\begin{aligned}\overline{\frac{z}{w}} &= \overline{\frac{a+bi}{c+di}} = \overline{\frac{(a+bi)(c-di)}{(c+di)(c-di)}} = \overline{\frac{ac-adi+bci-bdi^2}{c^2-cdi+cdi-d^2i^2}} = \overline{\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}} \\ &= \frac{(ac+bd)-(bc-ad)i}{c^2+d^2} = \frac{ac+adi-bci-bdi^2}{c^2+d^2} = \frac{(a-bi)(c+di)}{(c-di)(c+di)} = \frac{a-bi}{c-di} = \frac{\overline{z}}{\overline{w}},\end{aligned}$$

$$\overline{\overline{z}} = \overline{a-bi} = a+bi = z,$$

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\begin{aligned}
|zw|^2 &= |(a+bi)(c+di)|^2 = |(ac-bd)+(ad+bc)i|^2 = (ac-bd)^2+(ad+bc)^2 \\
&= a^2c^2 - 2acbd + b^2d^2 + a^2d^2 + 2adbc + b^2c^2 = a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \\
&= \big(a^2+b^2\big)\big(c^2+d^2\big) = |z|^2|w|^2, \\
|zw| &= |z||w|, \\
|z| &= \left|\frac{z}{w}\cdot w\right| = \left|\frac{z}{w}\right||w|, \\
\left|\frac{z}{w}\right| &= \frac{|z|}{|w|}, \\
zw &= (a+bi)(c+di) \\
&= |z|\big(\cos(\theta_z)+\sin(\theta_z)i\big)|w|\big(\cos(\theta_w)+\sin(\theta_w)i\big) \\
&= |z||w|\big(\cos(\theta_z)\cos(\theta_w)+\cos(\theta_z)\sin(\theta_w)i+\sin(\theta_z)\cos(\theta_w)i+\sin(\theta_z)\sin(\theta_w)i^2\big) \\
&= |z||w|\Big[\big(\cos(\theta_z)\cos(\theta_w)-\sin(\theta_z)\sin(\theta_w)\big)+\big(\sin(\theta_z)\cos(\theta_w)+\cos(\theta_z)\sin(\theta_w)\big)i\Big] \\
&= |z||w|\big(\cos(\theta_z+\theta_w)+\sin(\theta_z+\theta_w)i\big), \\
\arg(zw) &= \arg(z)+\arg(w), \\
z &= \frac{z}{w}\cdot w, \\
\arg(z) &= \arg\!\left(\frac{z}{w}\right)+\arg(w), \\
\arg\!\left(\frac{z}{w}\right) &= \arg(z)-\arg(w).
\end{aligned}
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Hermed er samtlige påstande bevist. □